$(B_1+B_2+B_3+\ldots+B_n)2\pi=\int_{-\infty}^{\infty}\frac{x^{2m}dx}{1+x^{2n}}$, by a well known formula in Calculus.

$$\text{Hence, } \sum_{x=-\infty}^{x=+\infty}\frac{x^{2m}}{1+x^{2n}}=\frac{\pi}{n\sin\frac{(2m+1)\pi}{2n}}=2\sum_{x=-0}^{x=+\infty}\frac{x^{2m}}{1+x^{2n}}.$$

$$\text{Hence, } \sum_{x=0}^{x=+\infty}\frac{x^{2m}}{1+x^{2n}}=\frac{\pi}{2n\sin\frac{(2m+1)\pi}{2n}}.$$

Also solved by J. I. Wodo.

---

## NUMBER THEORY AND DIOPHANTINE ANALYSIS.

---

**120. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.**

Find the prime numbers $p$ for which $x^2-pxz-px-z+p^2-3=0$ has more than two sets of positive integral solutions $x$, $z$, each $<p$.

**II. Solution by E. B. ESCOTT, Ann Arbor, Mich.**

The solution of this problem which was published in the March number of THE AMERICAN MATHEMETICAL MONTHLY does not seem to answer the question exactly, but rather proposes instead another problem which seems equally difficult, as there is no certainty that the numbers $p$ so found will be prime. Thus, the solution given leads to the following values of $p$, 74 and 394199, neither of which is prime.

The following method will give an indefinite number of primes $p$ which satisfy the given equation, and leads to formulas such as $p=x^2-x-1$;

$$x=n^2+n-1,\ p=n^3+n^2-2n-1. \tag{1}$$

$$\text{In the equation, } z+1=n=\frac{x^2+p^2-2}{px+1}, \tag{2}$$

$p$ and $x$ occur symmetrically, and $n$ will be unchanged if $p$ and $x$ are interchanged. If the roots of the equation (2), considered as an equation in $x$, are $x_1$ and $x_2$, we have $x_2=p_1n-x_1$, and since, as just noticed, $p$ and $x$ may be interchanged, we may take this value of $x_2$ for a new $p$ (say $p_2$) and take the new $x_1=p_1$. $n$ will be unchanged. The new $x_2=p_2n-x_1=p_2n-p_1$. Let this equal $p_3$, etc. Then we see that in the recurring series

$P_n$ : $x_1$, $p_1$, $p_1n-x_1$, ..., where the scale of relation is $P_{n+2}=nP_{n+1}-P_n$,

any two adjacent terms of the series may be taken as $x$ and $p$, the initial terms $x_1$ and $p_1$ being any solution of the equation (2).

Example. Let $x_1=1$, $p_1=a$, then $n=a-1$. The series is

$$1,\ a,\ a^2-a-1,\ a^3-2a^2-a+1,\ a^4-3a^3+3a,\ a^5-4a^4+2a^3+5a^2-2a-1.$$

If we take the second and third terms for $x$ and $p$, we shall have the first of the formulas given; with the third and fourth terms, we shall have the second formulas (1).

*Numerical Examples.*

| | |
|---|---|
| $n=2$ | 1, 3, 5, 7, 9, 11, 13, ... |
| $n=3$ | 1, 4, 11, 29, 76, 199, 521, ... |
| $n=4$ | 1, 5, 19, 71, 265, 989, 3691, ... |
| $n=5$ | 1, 6, 29, 139, 666, ... |
| $n=6$ | 1, 7, 41, 239, 1393, ... |

Every prime occurring in this table is a value of $p$ and the term preceding it is the corresponding value of $x$. Examples of primes $p$ with four values of $x$: $p=29$, $x=1, 6, 11, 27$; $p=71$, $x=1, 9, 19, 69$; $p=239$, $x=1, 16, 41, 237$.

**III. Solution by the PROPOSER.**

The following method, which is that developed by the Proposer at the time of setting the problem, will be shown to lead to all possible solutions. If $(p^2-1)^2$ has the factor $1+xp$, the complementary factor has the form $1+yp$ and there exists an integer $k\geqq 1$ (designated $z+1$ in the proposed congruence) such that

$$x+y=pk,\quad xy+k=p^2-2.$$

Eliminating $y$ and $x$ in turn, we get

$$k=\frac{x^2+p^2-2}{1+xp},\quad k=\frac{y^2+p^2-2}{1+yp}\ldots(1),$$

$$(p^2-1)^2=(1+xp)[1+(pk-x)p]=(1+yp)[1+(pk-y)p]\ldots(2).$$

In $(1_1)$, $x$ and $p$ enter symmetrically; in $(1_2)$, $y$ and $p$. Hence

$$(x^2-1)^2=(1+px)[1+(xk-p)x],\quad (y^2-1)^2=(1+py)[1+(yk-p)y]\ldots(3).$$

If $k=1$ in (1), then from $(x-p)^2\geqq 0$ we get $p\leqq 2$. We assume that $p\geqq 2$, whence $k>1$.

Denote by the symbol $\{x, p\}$ a pair of numbers such that
($i$) $x$ and $p$ are positive integers;
($ii$) $x<p$;
($iii$) $1+xp$ is a factor of $(p^2-1)^2$.
From one such pair we may derive a right-neighboring pair $\{p, y\}$, where $y=pk-x$. Since $x<p$, $k>1$, we have $p<y$, so that properties ($i$) and ($ii$) hold for $\{p, y\}$. Property ($iii$) is true by $(3_2)$. Hence any pair leads to a chain of right-neighboring pairs:

$$\{x, p\},\ \{p, pk-x\},\ \{pk-x,\ (pk-x)k-p\}\ldots(4).$$

For $x=1$, $k=p-1$, by (1). Hence we have the successive pairs

$$\{1, k+1\},\ \{k+1, k^2+k+1\},\ \{k^2+k-1,\ k^3+k^2-2k-1\}\ldots(5).$$

When the second member of a pair (5) is a prime $p$, the first member is a solution $x$ $(x<p)$. We proceed to show that every pair $\{X, P\}$ leading to a solution and having $P$ prime may be obtained from the pairs (5) by assigning a suitable value to $k$. To this end we consider the pair $\{w, x\}$, $w=xk-p$, which is left-neighboring to a given pair $\{x, p\}$. We determine the conditions under which $\{w, x\}$ has the properties ($i$)—($iii$). The third property holds in view of $(3_1)$. By the latter,

$$wx=A-1,\ A\equiv\frac{(x^2-1)^2}{1+px},\ w=\frac{x^3-2x-p}{1+px}<\frac{x^3}{x^2},$$

since $x<p$. Hence $w<x$, and property ($ii$) holds. Finally, ($i$) holds, viz., $w$ is positive, unless $A=0$ or 1. Hence we may form in succession left-neighboring pairs until we reach a pair with $A=0$ or 1. One of the latter cases must ultimately present itself, since a series of decreasing positive integers must terminate. If $A=0$, then $x=1$, and the chain contains the first pair (5), so that $\{X, P\}$ occurs in the list (5). If $A=1$, then $p=xk$, $k=x^2-2$. The first terms of the pairs (4) are then

$$x,\ x(x^2-2),\ x(x^2-2)^2-x,\ x(x^2-2)^3-2x(x^2-2),\ \ldots$$

a series of increasing integers with the factor $x$, so that no one is a prime $P$.

## MECHANICS.

**210. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.**

A rigid triangle is formed of three weightless, smoothly jointed, rigid rods $BC$, $CA$, $AB$. At their mid points $D$, $E$, $F$, respectively, are small, smooth rings, through which passes an endless, stretched, elastic string, forming the triangle $DEF$. Find by graphical construction the reaction at the joints.